

Characterization of Class of Measurable

Borel Lattices

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Abstract

In this paper we introduce the concept of measurable Borel lattices, σ - lattice and δ - lattice. We prove that class of Borel lattices is measurable and first valuation theorem, second valuation theorem. Also we prove that L_σ , L_δ lattices are Borel lattices.

Keywords: Open sub lattice, open lattice, closed lattice, σ - lattice and δ - lattice

§1. INTRODUCTION

The notation of a length of a sub lattice in the lattice theory signals out a special type of Partially ordered set for detailed investigation. To make such a definition worthwhile, it must be shown that this class of Partially ordered set is a very useful class in various branches of Mathematics namely Analysis, Topology, Logic, Algebra etc[2]. The study of these Partially ordered set is generalized with the help of measure

theory. Thus the lattice measure m on the σ – algebra $\sigma(L)$ have the following properties[3].

(i) For an interval J , $m(J) = l(J)$.

(ii) If $\{E_k\}$ be sequence of disjoint lattices (for which m is defined), $m(\bigvee_{k=1}^{\infty} E_k) \leq$

$\sum_{k=1}^{\infty} m(E_k)$ and $m(\bigvee_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k)$ (Countable sub-additive and countably additive).

By the length of a chain consisting of r elements (that is, being of the form $x_0 < x_1 < \dots < x_{r-1}$) we shall mean the non negative integer $r - 1$ and length of a chain consisting of an infinite number of elements will be symbolized by ∞ . Then the length of a Partially ordered set P is defined as the length of least upper bounds of the lengths of all sub chains in P . Depending on the above considerations, P is said to be either finite or infinite length[4]. So we consider a lattice of infinite length is measurable (Since measure of an interval is its length).

In section2, according to[5] the definition of a lattice σ – algebra $\sigma(L)$, lattice measure on the lattice σ – algebra $\sigma(L)$ and by[4] the definition of length of a chain, length of a Partially ordered set. And by[3] Result2.1.

In section3, we prove every σ – lattice and δ - lattice are measurable. And we prove first and second valuation theorems.

In section4, we prove an important lemma (a, ∞) is measurable lattice and we introduce a concept of Borel lattices. About Borel lattices for every class A of sub lattices of a lattice L there always exists a smallest σ – algebra β , consisting of a class A . If $\{B_\alpha, \alpha \in \Delta\}$, is a collection of all σ – algebra containing A , then $\bigwedge_{\alpha \in \Delta} B_\alpha = \beta$ is the smallest σ – algebra containing A . This σ – algebra β is called σ – algebra generated by A . Here we define a class of Borel lattice, definitions of L_σ , L_δ , $L_{\sigma\delta}$ and $L_{\delta\sigma}$ lattices and the most important theorem of every Borel lattice is measurable lattice. Also the class of Borel lattice contains both closed lattices and open lattices. And the Results L_σ , L_δ lattices are Borel lattices.

§2. PRELIMINARIES

In this paper, we shall consider the union and intersection of set theory as Λ and \vee

In this section, we shall briefly review the well-known facts about lattice theory (e.g. Birkhoff [1]), propose an extension lattice, and investigate its properties. (L, Λ, \vee) is called a lattice if it is enclosed under operations \wedge and \vee and satisfies, for any elements x, y, z , in L :

(L1) the commutative law: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

(L2) the associative law:

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \text{ and } x \vee (y \vee z) = (x \vee y) \vee z.$$

(L3) the absorption law: $x \vee (y \wedge x) = x$ and $x \wedge (y \vee x) = x$.

Hereafter, the lattice (L, \wedge, \vee) will often be written as L for simplicity.

A lattice (L, \wedge, \vee) is called distributive if, for any x, y, z , in L .

(L4) the distributive law holds:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \text{ and } x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A lattice L is called complete if, for any subset A of L , L contains the supremum $\vee A$ and the infimum $\wedge A$ [1].

Definition2.1. Unless otherwise stated, X is the entire set and L is a lattice of any subsets of X . If a lattice L satisfies the following conditions, then it is called a lattice σ -Algebra;

(1) $\forall h \in L, h^c \in L$

(2) if $h_n \in L$ for $n = 1, 2, 3, \dots$, then $\bigvee_{n=1}^{\infty} h_n \in L$.

We denote $\sigma(L)$, as the lattice σ -Algebra generated by L .

Definition2.2. The ordered pair $(X, \sigma(L))$ is said to be lattice measurable space

Definition2.3. If $m: \sigma(L) \rightarrow \mathbb{R} \cup \{\infty\}$ satisfies the following properties, then m is called a lattice measure on the lattice σ -Algebra $\sigma(L)$.

(1) $m(\phi) = m(0) = 0$.

(2) $\forall h, g \in \sigma(L)$, s.t. $m(h), m(g) \geq 0$; $h \leq g \Rightarrow m(h) \leq m(g)$.

(3) $\forall h, g \in \sigma(L)$: $m(h \vee g) + m(h \wedge g) = m(h) + m(g)$.

(4) If $h_n \in \sigma(L)$, $n \in \mathbb{N}$ such that $h_1 \leq h_2 \leq \dots \leq h_n \leq \dots$, then $m(\bigvee_{n=1}^{\infty} h_n) = \lim m(h_n)$.

Let m_1 and m_2 be lattice measures defines on the same lattice σ -Algebra $\sigma(L)$. If one of them is finite, the set function $m(E) = m_1(E) - m_2(E)$, $E \in \sigma(L)$ is well defined and countably additive on $\sigma(L)$. However, it is necessarily nonnegative; it is called a signed lattice measure.

Definition2.4. A set A is said to be lattice measurable set or measurable lattice, if A belongs to $\sigma(L)$.

Definition2.5. For any a, b belongs to L , we define an open sub lattice by $(a, b) = \{x \mid a < x < b\}$.

Definition2.6. Union of countable number of open sub lattices is called an open lattice.

Definition2.7. For any a, b belongs to L , we define a closed sub lattice by $[a, b] = \{x \mid a \leq x \leq b\}$.

Definition2.8. A lattice is said to be closed lattice if its complement is open lattice.

Definition2.9. σ – lattice: Countable union of measurable lattices.

Definition2.10. δ - lattice: Countable intersection of measurable lattices.

Definition2.11. Let $a_0 < a_1 < a_2 \dots \dots \dots < a_n$ be a finite chain in P , we define the length of the chain as n and the length of an infinite chain will be defined as ∞ .

Definition2.12. The length of a partially ordered set P will be defined as the least upper bound of the lengths of all sub chains in P . Note that the length of P . O. Set will be ∞ .

Result2.1. If E is measurable set iff E^c is also measurable.

§3. σ - lattice and δ - lattice:

Theorem3.1. If E_1, E_2, \dots are pair wise disjoint measurable lattices and $E = \bigvee_{k=1}^{\infty} E_k$,

then E is measurable (or) Every σ - lattice is measurable and also $m(E) = \sum_{k=1}^{\infty} m(E_k)$

Proof. Part1. It is given that $E_i \wedge E_j = \phi$ for $i \neq j$, we have $m(\bigvee_{k=1}^{\infty} E_k) \leq$

$$\sum_{k=1}^{\infty} m(E_k) \dots \dots \dots (1)$$

Clearly $\bigvee_{k=1}^{\infty} E_k \geq \bigvee_{k=1}^n E_k$ implies $m(\bigvee_{k=1}^{\infty} E_k) \geq m(\bigvee_{k=1}^n E_k) \dots \dots \dots (2)$

We know that if $E_i \wedge E_j = \phi$ implies $m(E_1 \vee E_2) = m(E_1) + m(E_2)$ (from definition 2.3.)

Extending, by induction, the result for infinite number of pair wise disjoint lattices, we get

$$m(\bigvee_{k=1}^n E_k) = \sum_{k=1}^n m(E_k) \text{ implies by(2) we have } m(\bigvee_{k=1}^{\infty} E_k) \geq \sum_{k=1}^n m(E_k)$$

letting $n \rightarrow \infty$, $m(\bigvee_{k=1}^{\infty} E_k) \geq \sum_{k=1}^{\infty} m(E_k) \dots\dots\dots(3)$

From (1) and (3) $m(\bigvee_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k)$.

Part2. Let E_1, E_2, \dots are pair wise disjoint measurable lattices

Clearly $E = \bigvee_{k=1}^{\infty} E_k = E_1 \vee (E_2 \wedge E_1^c) \vee \dots \vee (E_k \wedge (\bigvee_{k=1}^{n-1} E_k^c)) \vee \dots$

Evidently the lattices $E_1, E_2 \wedge E_1^c, \dots$ are disjoint measurable lattices and hence by part1 $\bigvee_{k=1}^{\infty} E_k$ is measurable lattice hence every σ - lattice is measurable.

Theorem 3.2. First valuation theorem.

Suppose that $\{E_k\}$ is monotonic increasing sequence of measurable lattices and $E = \bigvee_{k=1}^{\infty} E_k$ then

$$m(E) = \text{Lt}_{n \rightarrow \infty} m(E_n).$$

Proof. Write $E = E_1 \vee (E_2 \wedge E_1^c) \vee \dots \vee (E_k \wedge (\bigvee_{k=1}^{n-1} E_k^c)) \vee \dots$

that is $E = E_1 \vee (\bigvee_{k=1}^{\infty} (E_{k+1} \wedge E_k^c))$ (A Disjoint union)

$$\begin{aligned} \text{By theorem 3.1. } m(E) &= m(E_1) + \sum_{k=1}^{\infty} m(E_{k+1} - E_k) = m(E_1) + \sum_{k=1}^{\infty} (m(E_{k+1}) - m(E_k)) \\ &= m(E_1) + \text{Lt}_{n \rightarrow \infty} \sum_{k=1}^n (m(E_{k+1}) - m(E_k)) \\ &= m(E_1) + \text{Lt}_{n \rightarrow \infty} [m(E_2) - m(E_1) + \dots + m(E_n) - m(E_{n-1})] \\ &= m(E_1) + \text{Lt}_{n \rightarrow \infty} [-m(E_1) + m(E_n)] = m(E_1) - m(E_1) + \text{Lt}_{n \rightarrow \infty} m(E_n) = \text{Lt}_{n \rightarrow \infty} m(E_n) \end{aligned}$$

Theorem 3.3. If E_1, E_2, \dots are measurable lattices then $\bigwedge_{k=1}^{\infty} E_k$ is measurable lattice (or) every δ - lattice is measurable.

Proof. By theorem 3.1. $E = \bigvee_{k=1}^{\infty} E_k$ is measurable lattice.

Let $G = \bigwedge_{k=1}^{\infty} E_k$, then $G^c = (\bigwedge_{k=1}^{\infty} E_k)^c = \bigvee_{k=1}^{\infty} E_k^c$

Given that each E_k is measurable lattice implies by Result 2.1. each E_k^c is measurable lattice implies $\bigvee_{k=1}^{\infty} E_k^c$ is measurable lattice. (Every σ - lattice is measurable) implies G^c is measurable lattice implies G is measurable lattice. (Result 2.1.).

Theorem 3.4. Second valuation theorem

Suppose that $\{E_k\}$ is a monotonic decreasing sequence of measurable lattices and $E = \bigwedge_{k=1}^{\infty} E_k$, then $m(E) = \text{Lt}_{n \rightarrow \infty} m(E_n)$.

Proof. Let $E = \bigwedge_{k=1}^{\infty} E_k$, evidently $E_1 = E \vee (E_1 \wedge E_2^c) \vee (E_2 \wedge E_3^c) \vee \dots$

$$\begin{aligned} \text{then } m(E_1) &= m(E) + \sum_{k=1}^{\infty} (m(E_k) - m(E_{k+1})) = m(E) + \text{Lt}_{n \rightarrow \infty} \sum_{k=1}^n (m(E_k) - m(E_{k+1})) \\ &= m(E) + \text{Lt}_{n \rightarrow \infty} [m(E_1) - m(E_2) + \dots + m(E_n) - m(E_{n+1})] = m(E) + \\ &\quad \text{Lt}_{n \rightarrow \infty} [m(E_1) - m(E_{n+1})] \\ &= m(E) + m(E_1) - \text{Lt}_{n \rightarrow \infty} m(E_{n+1}) \text{ implies } m(E) = \text{Lt}_{n \rightarrow \infty} m(E_n). \end{aligned}$$

§4. CLASS OF BOREL LATTICES:

Lemma 4.1. The interval (a, ∞) is measurable lattice.

Proof. For $(a, \infty) = \bigvee_{n \in \mathbb{N}} (a, n) =$ a countable union of sub lattices.

Here $\{(a, n)\}$ is a monotonically increasing sequence hence by Theorem 3.2.

$$m(a, \infty) = m(\bigvee_{n \in \mathbb{N}} (a, n)) = \text{Lt}_{n \rightarrow \infty} m(a, n) = \text{Lt}_{n \rightarrow \infty} (n - a) = \infty.$$

Definition 4.1. Class of Borel lattices: The smallest σ - algebra containing class of all open lattices is called class of Borel lattices in \mathbb{R} and it is denoted by β . Every member of Borel class β are called Borel lattice of \mathbb{R} .

Notation 4.1. The family of measurable lattices is denoted by μ .

Theorem 4.1. Every Borel lattice is measurable lattice.

Proof. We know that measure of an interval is its length. By Lemma 4.1. for any a belongs to \mathbb{R} , (a, ∞) is measurable lattice and hence its complement $(-\infty, a]$ is also measurable lattice. Now for any b belongs to \mathbb{R} , $(-\infty, b) = \bigvee_{n=1}^{\infty} (-\infty, b - \frac{1}{n}]$ and since

$(-\infty, b - \frac{1}{n}]$ is measurable lattice so is $(-\infty, b)$. Since $(a, b) = (-\infty, b) \wedge (a, \infty)$, it follows that each open sub lattice is measurable lattice. Also we know that an open lattice is the union of countable number of open sub lattices we get every open lattice is a measurable lattice, but β is the smallest σ - algebra containing class of all open lattices in R . that is $\beta \leq \mu$ implies every Borel lattice is a measurable lattice.

Note 4.1. Every Borel lattice is measurable lattice but converse is not true.

Theorem 4.2. The class of borel lattices is the smallest σ - algebra β containing class of all closed lattices in R .

Proof. By Definition 4.1. the class of Borel lattices is smallest σ - algebra β containing class of all open lattices in R . Let \odot be the class of open lattices in R and C be the class of closed lattices in R . Let $A < C$ implies $A^c < \odot$ implies $A^c < \beta$ implies $A < \beta$ (Since β is a σ - algebra). Therefore $C < \beta$. Let D be the σ - algebra containing class of all closed lattices in R . Let $A < \odot$ implies $A^c < C$ implies $A^c < D$ implies $A < D$ (Since D is σ - algebra). Therefore $\odot < D$. Evidently D is a σ - algebra containing class of closed lattices as well as class of open lattices. But β is the smallest σ - algebra containing class of open lattices in R implies $\beta < D$. Therefore β is the smallest σ - algebra containing class of all closed lattices in R .

Definition 4.2. L_σ - lattice: A lattice which is countable union of closed lattices

Definition 4.3. L_δ - lattice: A lattice which is countable intersection of open lattices.

Result 4.1. Every open lattice is a L_σ - lattice.

Proof. We know that an open lattice is the union of countable number of open sub lattices. By Theorem 4.2. every open sub lattice is a closed lattice. Hence an open lattice becomes countable union of closed lattices. Therefore every open lattice is L_σ - lattice.

Result 4.2. L_σ - lattice and L_δ - lattice are Borel lattices.

Proof. By Theorem 4.2. all closed lattices are present in a class of Borel lattices. Implies countable union of closed lattices are also present in a class of Borel lattices implies L_σ - lattices are present in a class of Borel lattices. Therefore every L_σ - lattice is a Borel lattice. Similarly by the Definition 4.1. all open lattices are present in a class of Borel lattices implies countable intersection of open lattices are also present

in a class of Borel lattices implies L_δ - lattices are present in a class of Borel lattices. Therefore every L_δ - lattice is a Borel lattice.

Definition 4.4. $L_{\sigma\delta}$ - lattice: Countable intersection of L_σ - lattice.

Definition 4.5. $L_{\delta\sigma}$ - lattice: Countable union of L_δ - lattice.

Result 4.3. $L_{\sigma\delta}$ - lattice and $L_{\delta\sigma}$ - lattice are Borel lattices.

Proof. Proof is similar to Result 4.2.

REFERENCES

- [1] Birkhoff. G, Lattice Theory 3rd ed., AMS Colloquim Publications, Providence, RI, 1967.
- [2] Gratzler. G, General Lattice Theory, Academic Press Inc., 1978.
- [3] Royden. H.L., Real Analysis, 3rd ed., Macmillan Publishing, New York, 1981.
- [4] Szasz Gabor, Introduction to lattice theory, academic press, New York and London.
- [5] Tanaka. J, Hahn Decomposition Theorem of Signed Lattice Measure, arXiv:0906.0147V011 [Math.CA] 31 May 2009.

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