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# A Unique Approach on Upper Bounds for the Chromatic Number of Total Graphs

[1]J. Venkateswara Rao and [2]R.V.N. Srinivasa Rao
[1]Department of Mathematics, Mekelle University Main Campus, Mekelle, Ethiopia
[2]Department of Mathematics, Guntur Engineering College, Guntur, A.P, India

*Corresponding Author: J. Venkateswara Rao, Department of Mathematics, Mekelle University Main Campus, Mekelle, Ethiopia*

## ABSTRACT

This study is an investigation on upper bound for the chromatic number of a graph. In this study, it has been extended the concept of an upper bound for the chromatic number of a graph to total graphs. Further proved that the chromatic number, $\chi(T(G))\leq[s/s+1(\Delta(T(G)+2))]$ for any graph G where $\Delta(T(G))$ is maximum degree in T(G), s be the maximum number of vertices with same degree.

**Key words:** Total graph, chromatic number, k-degenerate graphs, degree of a graph, upper bound

## INTRODUCTION

Determining chromatic number $\chi(G)$ is an old and hard problem. A classical result of Brooks (1941) says that $\chi(G)\leq\Delta(G)+1$, where, $\Delta(G)$ is the maximum degree in G. In addition, Brooks (1941) showed that the complete graphs and odd cycles are the only graphs for which the upper bound attained. Excluding the existence of smaller complete sub graphs can further improve the upper bounds for $\chi(G)$, as it can be seen from the result obtained independently by Borodin and Kostochka (1977), Catlin (1978) and Lawrence (1978). Stacho (2002) has made a note on upper bound for chromatic number of a graph. Praroopa and Rao (2011a) established a Lattice in Pre A*-Algebra. Praroopa and Rao (2011b) obtained Pre A*-Algebra as a Semilattice. Rao and Satyanarayana (2010) made a Semilattice Structure on Pre A*-Algebra. Rao and Kumar (2010). contributed the structure of Weakly Distributive and Sectionally *Semilattice. Praroopa and Rao (2011b) obtained Logic Circuits and Gates in Pre A*-Algebra. Rao and Rao (2010) desrived Subdirect Representations in A*-Algebras. Satyanarayana *et al.* (2011) obtained Prime and Maximal Ideals of Pre A*-Algebra.

Recently, Pramada *et al.* (2012) made a characterization of complex integrable lattice functions and mu-free lattices.

Most upper bounds on the chromatic number come from algorithm that produces colouring. For example assign colours to the vertices yields $\chi(G)\leq n(G)$. This bound is best possible, since, $\chi(K_n) = n$, but it holds with equality only for complete graphs. We can improve best possible bounds by obtaining another bound that is always at least as good for example, $\chi(G)\leq n(G)$ uses nothing about the structure of G, we can do better by coloring the vertices in some order and always using the least available colours.

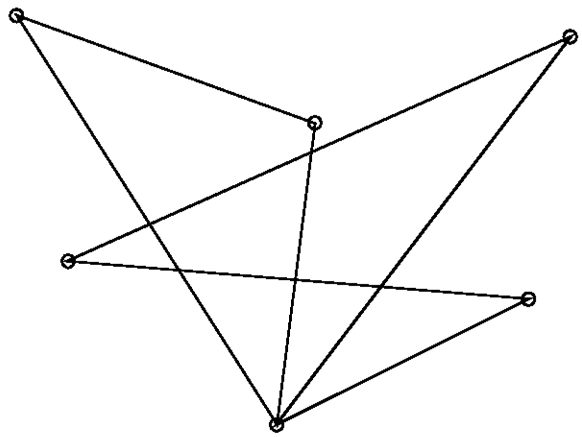

Fig. 1: The exam scheduling problem

## TERMINOLOGY AND PRELIMINARIES

In this study, we consider only finite graphs. The terminology and notation not presented here can be found by West (1996).

**Definition 1 (Clark and Holton, 1991):** The minimum number of colours needed to color the vertices of a graph G so that no two adjacent vertices are the same is called the chromatic number and is written by $\chi(G)$. For example, if G is the 3-cycle or triangle graph, then $\chi(G) = 3$

**Note 1:** Figure 1 represents the exam scheduling problem. Each vertex stands for a course. An edge between two vertices indicates that a student is taking both courses and therefore the exams cannot be scheduled at the same time. Each exam time slot is associated with a color. A schedule that creates no conflicts for any student corresponds to a coloring of the vertices such that no adjacent vertices receive the same colour.

**Definition 2:** Let G be a graph with vertex set V(G) and edge set E(G). The total graph of G, denoted by T(G) is defined in the following way. The vertex set of T(G) is V(G) union E(G). Two vertices x, y in the vertex set of T(G) are adjacent in T(G) in case one of the following holds:

(i) x, y are in V(G) and x is adjacent to y in G
(ii) x, y are in E(G) and x, y are incident in G
(iii) x is in V(G), y is in E(G) and x, y are incident in G

**Note 2:** By the condition (i) of definition 2, we can conclude that G is a sub graph of T(G).

**Example 1:** Consider the graph G given in Fig. 2. It is the cycle with three edges. We may denote it by $C_3$:

$$V(G) = \{v_1, v_2, v_3\};\ E(G) = \{e, f, g\}$$

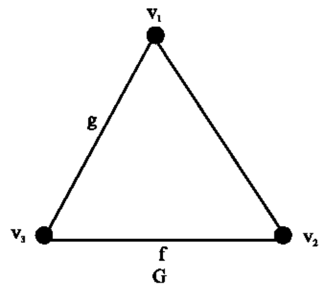


Fig. 2: Graph of G

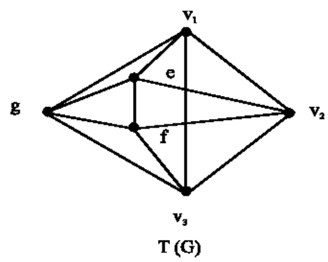


Fig. 3: Graph of T(G)

Let us construct the total graph T (G) of G:

- $V(T(G)) = V(G) \cup E(G) = \{v_1, v_2, v_3, e, f, g\}$
- Since, G is a sub graph of T(G), it follows that e, f, g $\in$ E (T(G))
- Since e, f $\in$ E(G) are adjacent in G, $\overline{ef} \in E(T(G))$
- Since f, g $\in$ E(G) are adjacent in G, $\overline{fg} \in E(T(G))$
- Since g, e $\in$ E(G) are adjacent in G, $\overline{ge} \in E(T(G))$
- Since $v_1 \in V(G)$, $e \in E(G)$ are adjacent in G, $\overline{v_1e} \in E(T(G))$
- Since $v_2 \in V(G)$, $e \in E(G)$ are adjacent in G, $\overline{v_2e} \in E(T(G))$
- Since $v_2 \in V(G)$, $f \in E(G)$ are adjacent in G, $\overline{v_2f} \in E(T(G))$
- Since $v_3 \in V(G)$, $f \in E(G)$ are adjacent in G, $\overline{v_3f} \in E(T(G))$
- Since $v_3 \in V(G)$, $g \in E(G)$ are adjacent in G, $\overline{v_3g} \in E(T(G))$
- Since $v_1 \in V(G)$, $g \in E(G)$ are adjacent in G, $\overline{v_1g} \in E(T(G))$
- Therefore, $E(T(G)) = \{e, f, g\ \overline{ef}, \overline{fg}, \overline{ge}, \overline{v_1e}, \overline{v_2e}, \overline{v_2f}, \overline{v_2f}, \overline{v_3g}, \overline{v_1g}\}$

The graph of T(G) is given by Fig. 3.

**Definition 3:** In graph theory, a k-degenerate graph is an undirected graph in which every sub graph has a vertex of degree at most k, that is, some vertex in the sub graph touches k or fewer of the sub graph's edges. The degeneracy of a graph is the smallest value of k for which it is k-degenerate. The degeneracy of a graph is a measure of how sparse it is and is within a constant factor of other sparsity measures such as the arbor city of a graph.

**Result 1: Greedy coloruing:** The greedy colouring relative to a vertex ordering $v_1,.......v_n$ of V(G) is obtained by colouring vertices in the order $v_1,.......v_n$, assigning to $v_i$ the smallest-indexed colour not already used on its lower indexed neighbors.

**Result 2:** For any graph G, $\chi(G)\leq\Delta(G)+1$.

**Proof:** In a vertex ordering, each vertex has at most $\Delta(G)$ earlier neighbors, so the greedy coloring cannot be forced to use more than $\Delta(G)+1$ colors. This proves constructively $\chi(G)\leq\Delta(G)+1$.

**Result 3:** If a graph G has degree sequence $d_1\geq d_2\geq d_3\geq ..........d_n$, then, $\chi(G)\leq 1+\text{Max}_i \min\{d_i, i\text{-}1\}$.

**Proof:** We apply greedy colouring to the vertices in non-increasing order of degree. When, we colour the ith vertex $v_i$, it has at most $\min\{d_i, i\text{-}1\}$ earlier neighbors, so at most this may colours appear on its earlier neighbors. Hence, the color we assign to $v_i$ is at most $1+\min\{d_i, i\text{-}1\}$.This holds for each vertex, so we maximize over i to obtain the upper bound on the maximum colour used.

**Note 3:** The bound in the result 3 is always at most $1+\Delta(G)$, so, this is always at least as good as result 2.

## UPPER BOUND OF THE CHROMATIC NUMBER OF TOTAL GRAPHS

The existence of smaller complete sub graphs can further improve the upper bounds for $\chi(G)$, as it can be seen from the result obtained independently by Borodin and Kostochka (1977), Catlin (1978) and Lawrence (1978). We extend the result to total graph T(G) for any graph G.

**Proposition 1:** If $K_r\not\subset T(G)$ for a total graph of any graph G, where, $4\leq r\leq\Delta(T(G))+1$, then:

$$\chi(T(G))\leq\frac{r-1}{r}(\Delta(T(G))+2)$$

This result is in fact nice application of Brooks theorem and the result the following result, observed by Lovasz (1996): if $d_1+d_2+d_3+..........d_q\geq\Delta(T(G))\text{-}q+1$, then V(T(G)) can be decomposed in to classes $V_1, V_2,......., V_q$, such that the subgraph $G_i$ induced by $V_i$ has $\Delta(G_i)\leq d_i$. Letting $q=\lfloor(\Delta(T(G))+1)/r\rfloor$, $d_1 = d_2 = ........d_{q-1} = r\text{-}1$ and $d_q\geq r\text{-}1$ so that $\Sigma d_i=\Delta(T(G))\text{-}q+1$ give the upper bound.

If a graph has only small complete sub graphs, then proposition 1 substantially improves Brooks upper bound. However, if the graph is dense, then it usually has large complete sub graphs and hence, the upper bound from proposition 1 is almost the same as the original Brooks upper bound. In what follows, we give another relaxation of Brooks (1941) theorem based on the following invariant. Let $V_i$ denote the set of vertices of degree i in the graph T(G). Now, we define $s = \max_{i\geq(\Delta(T(G))+2)/2}|V_i|$, i.e., s is the maximum number of vertices of the same degree, each at least $(\Delta(T(G))+2/2)$.

**Proposition 2:** For any graph G, $\chi(T(G))\leq\lceil s/s+1(\Delta(T(G)+2)\rceil$.

**Proof:** Let $d_1 \geq d_2 \geq d_3 \geq \ldots\ldots\ldots \geq d_n$ be the degree sequence of T(G). We let $k = \lceil s/s+1(\Delta(T(G)+2) \rceil$. We claim that $d_k<k$. If:

$$d_k < \frac{\Delta(T(G))+2}{2}$$

then since, $s \geq 1$, the claim is true. Otherwise:

$$d_k \geq \frac{\Delta(T(G))+2}{2}$$

Now for i = 1, 2, 3,... .k:

$$d_i \leq \Delta(T(G)) - \left\lceil \frac{i}{s} \right\rceil + 1 < \Delta(T(G)) - (\frac{i}{s} - 1) + 1 = \Delta(T(G)) - \frac{i}{s} + 2$$

In particular:

$$d_k < \Delta(T(G)) - \frac{k}{s} + 2 \leq k$$

as claimed.

It follows that T(G) is K-degenerate it is well known that vertices of any k-degenerate graph can be properly colored with at most k colours. Thus, we have:

$$\chi(T(G)) \leq \left\lceil \frac{s}{s+1}(\Delta(T(G)+2) \right\rceil$$

**Note 4:** We observed that in some cases Proposition 2 gives much better upper bound for $\chi(T(G))$ as proposition 1 does. Also note that proposition 2 does not use Brook's theorem.

**Results 4:** We determined that chromatic number of a total graph T(G) for any graph G,

$$\chi(T(G)) \leq \min\{\frac{r-1}{r}(\Delta(T(G))+2), \left\lceil \frac{s}{s+1}(\Delta(T(G))+2) \right\rceil\}$$

**Note 5:** The following example demonstrate the result obtained in results 4.

Consider the graph G given in Fig. 2 It is the cycle with three edges.

Figure 3 is the total graph T (G) of the graph G.

In Fig. 3 $\Delta(T(G)) = 4$.

S is the maximum no of vertices with same degree = 6

We find:

$$\left\lceil \frac{s}{s+1}(\Delta T((G))+2) \right\rceil = \left\lceil \frac{6}{6+1}(4+2) \right\rceil = \lceil 5.14 \rceil = 6$$

If $K_r \not\subset T(G)$ for a total graph of any graph G, where, $4 \le r \le \Delta(T(G)+1)$, $\Delta(T(G)) = 4$
Now:

$$\frac{r-1}{r}(\Delta(T(G))+2) = \frac{4-1}{4}(4+2) = 4.5$$

Hence:

$$\chi(T(G)) \le \min\{\frac{r-1}{r}(\Delta(T(G))+2), \left\lceil \frac{s}{s+1}(\Delta(T(G))+2) \right\rceil\}$$

$$\chi(T(G)) \le \min\{4.5, 6\}$$

The actual Chromatic number, $\chi(T(G))$ is 3 which is nearer to the minimum of the upper bounds obtained in propositions 1 and 2.

## CONCLUSION

This study has attained an upper bound for the chromatic number of total graphs and established that the chromatic number of total graph is:

$$\chi(T(G)) \le \left\lceil \frac{s}{s+1}(\Delta(T(G)+2) \right\rceil$$

for any graph G. Also, it has been provided a new dimension for the chromatic number of total graph. The chromatic number of total graph is:

$$\chi(T(G)) \le \min\{\frac{r-1}{r}(\Delta(T(G))+2), \left\lceil \frac{s}{s+1}(\Delta(T(G))+2) \right\rceil\}$$

## FUTURE EXPANSION

we may extend the concept of upper bounds for 1-quasi and 2-quasi total graphs.

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