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A CAYLEY THEOREM FOR A^*-ALGEBRAS

BY

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Abstract. This paper studies the Cayley's theorem for A^* -algebras Cayley's theorem for A^*-algebras is proved by showing first that an A^*-algebra structure can be defined on subsets A of ternary functions on a set X and then that every A^*-algebra is isomorphic with such a subalgebra of Ter (X) for a suitable set X.

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Introduction. E.G. Manes introduced an $Ada(A, \wedge, \vee, (-)^1, (-)_\pi, 0, 1, 2)$ in his paper [6], based on C-algebras introduced by Guzman and Squir in their paper [2]. P Koteswara Rao [3] introduced the concept of A^*-algebra, analogous to the Manes Adas [6].

The well known Cayley theorem for groups may be summarized as follows: Any group is isomorphic to a group of transformations on some set. Bloom, Esik and Manes [7] introduced a Cayley theorem for Boolean algebras, which says that any Boolean algebra is isomorphic to a Boolean algebra of binary functions on a set. In this paper we introduced a Cayley theorem for A^*-algebras which says that any A^*-algebra is isomorphic to an A^*-algebra of ternary functions on a set.

1. Preliminaries.

Definition 1.1. *An algebra* $(A, \wedge, *, (-)^\sim, (-)_\pi, 1)$ *is an* A^**-algebra if it satisfies: For* $a, b, c \in A$,

(i) $a_\pi \vee (a_\pi)^\sim = 1, (a_\pi)_\pi = a_\pi$ *where* $a \vee b = (a^\sim \wedge b^\sim)^\sim$;

(ii) $a_\pi \vee b_\pi = b_\pi \vee a_\pi$;
(iii) $(a_\pi \vee b_\pi) \vee c_\pi = a_\pi \vee (b_\pi \vee c_\pi)$;
(iv) $(a_\pi \wedge b_\pi) \vee (a_\pi \wedge (b_\pi)^\sim) = a_\pi$;
(v) $(a \wedge b)_\pi = a_\pi \wedge b_\pi, (a \wedge b)^\# = a^\# \vee b^\#$ *where* $a^\# = (a_\pi \vee a_\pi^\sim)^\sim$;
(vi) $a_\pi^\sim = (a_\pi \vee a^\#)^\sim, a^{\sim\#} = a^\#$;
(vii) $(a * b)_\pi = a_\pi, (a * b)^\# = (a_\pi)^\sim \wedge (b_\pi^\sim)^\sim$;
(viii) $a = b$ *if and only if* $a_\pi = b_\pi, a^\# = b^\#$.

We write 0 *for* $1^\sim, 2$ *for* $0 * 1$.

Example 1.2. $3 = \{0, 1, 2\}$ with the operations defined below is an A^*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$*$	0	1	2
0	0	2	2
1	1	1	1
2	0	2	2

x	0	1	2
$x^\sim$	1	0	2
x_π	0	1	0
$x^\#$	0	0	1

Note 1.3. From 1.1 (i) to 1.1 (iv) and by Huntington's theorem [1] we see that, $\mathcal{B}(A) = \{a_\pi / a \in A\}$ is a Boolean algebra with $\wedge, \vee, (-)^\sim, 0$ and $a \in \mathcal{B}(A) \Rightarrow a_\pi = a$. Since $1, 0, (a_\pi)^\sim \in \mathcal{B}(A)$, we have $1_\pi = 1, 0_\pi = 0, (a_\pi)_\pi^\sim = (a_\pi)^\sim$ and $a_\pi \wedge a^\# = 0, a * 0 = a_\pi$.

2. Cayley's theorem for A*-algebras

2.1. Cayley theorem for Groups. If G is a group, there is a set X such that G is isomorphic to a transformation group on X.

Bloom, Esik and Manes [7] have proved the following theorems 2.2, 2.3 which constitute a "Cayley theorem for Boolean algebras".

Theorem 2.2. Cayley theorem for Boolean algebras (part I): *Let B be a subset of $Bin(X)$ of all binary functions on a set X with the following properties:*

(i) B *contains* $\pi_1, \pi_1(x, y) = x$ *for all* $x, y \in X$;
(ii) $f(x, x) = x$ *for all* $x \in X$, *for all* $f \in B$;
(iii) $f(f(x, y), f(u, v)) = f(x, v)$ *for all* $f \in B, x, y, \mu, v \in X$;
(iv) $f(g(x, y), g(\mu, v)) = g(f(x, \mu), f(y, v))$ *for all* $f, g \in B$, *for all* $x, y, \mu, v \in X$ *and* B *is closed under the operations* $\wedge, \vee$ *and* $(-)'$;
(v) $(f \wedge g)(x, y) = f(g(x, y), y)$;
(vi) $(f \vee g)(x, y) = f(x, g(x, y))$;

(vii) $f'(x,y) = f(y,x)$ *for all* x,y *and for all* f. *Then* B *is a Boolean algebra. Such Boolean algebras on* X *are called "Guard Algebras" on* X.

Theorem 2.3. Cayley theorem for Boolean algebras (part II): *If* $(B, \wedge, \vee, (-)', 1, 0)$ *is any Boolean algebra, then there is a set* X *such that* B *is isomorphic to a Guard algebra on* X.

Theorem 2.4. *Let* A *be a subset of* $Ter(X)$ *of all ternary functions on* X *with the following properties:*

(i) $f(x,x,x) = x$ *for all* $f \in A, x \in X$;

(ii) $f(f(x_1,y_1,z_1), f(x_2,y_2,z_2), f(x_3,y_3,z_3)) = f(x_1,y_2,z_3)$ *for all* $f \in A$, *for all* $x_i, y_i, z_i \in X$;

(iii) $f(g(x_1,y_1,z_1), g(x_2,y_2,z_2), g(x_3,y_3,z_3)) = g(f(x_1,x_2,x_3), f(y_1,y_2,y_3), f(z_1,z_2,z_3))$, *for all* $f, g \in A$, *for all* $x_i, y_i, z_i \in X$;

(iv) A *contains the projections* π_1, π_2, π_3 *where* $\pi_1(x,y,z) = x, \pi_2(x,y,z) = y, \pi_3(x,y,z) = z$;

Further, A *is closed under the operations* $\wedge, \vee, (-)^{\sim}, (-)_{\pi}, *$ *defined by*

(v) $(f \wedge g)(x,y,z) = f(g(x,y,z), g(y,y,z), z)$;

(vi) $(f \vee g)(x,y,z) = f(g(x,x,z), g(x,y,z), z)$;

(vii) $f^{\sim}(x,y,z) = f(y,x,z)$;

(viii) $f_{\pi}(x,y,z) = f(x,y,y)$;

(ix) $(f * g)(x,y,z) = f(x, g(z,y,z), g(z,y,z))$, *for all* $f, g \in A$ *and for all* $x,y,z \in X$.

Then, A *is an* A^**-algebra.*

Such A^**-algebras are called Guard* - algebras on* X.

Proof. Define, $f^{\#}(x,y,z) = (f_{\pi} \vee f_{\pi}^{\sim})^{\sim}(x,y,z)$ for all $f \in A$ and for all $x,y,z \in X$. Now we prove for $f \in A$ that $f^{\#}(x,y,z) = f(y,y,x)$ for all $x,y,z \in X$. We have

$$\begin{aligned} f^{\#}(x,y,z) &= (f_{\pi} \vee f_{\pi}^{\sim})^{\sim}(x,y,z) = (f_{\pi}^{\sim} \wedge f_{\pi^{\sim}}^{\sim}(x,y,z) = \\ &= f_{\pi}^{\sim}(f_{\pi^{\sim}}^{\sim}(x,y,z), f_{\pi^{\sim}}^{\sim}(y,y,z), z)(By 2.4(v)) = \\ &= f_{\pi}^{\sim}(f_{\pi}^{\sim}(y,x,z), f_{\pi}^{\sim}(y,y,z), z)(By 2.4(vii)) = \\ &= f_{\pi}^{\sim}(f^{\sim}(y,x,x), f^{\sim}(y,y,y), z)(By 2.4 viii) = \\ &= f_{\pi}^{\sim}(f(x,y,x), f(y,y,y), z) = f_{\pi}(f(y,y,y), f(x,y,x), z) = \\ &= f(f(y,y,y), f(x,y,x), f(x,y,x)) = f(y,y,x)(By 2.4(ii)) \end{aligned}$$

Therefore, $f^{\#}(x,y,z) = f(y,y,x)$.

Proofs of the following are left to the reader.

For all $f, g, h \in A$,

$$f_\pi \vee f_\pi^\sim = \pi_1, (f_\pi)_\pi = f_\pi, f \vee g = (f^\sim \wedge g^\sim)^\sim; \tag{1}$$

$$f_\pi \vee g_\pi = g_\pi \vee f_\pi; \tag{2}$$

$$(f_\pi \vee g_\pi) \vee h_\pi = f_\pi \vee (g_\pi \vee h_\pi); \tag{3}$$

$$(f_\pi \wedge g_\pi) \vee (f_\pi \wedge g_\pi^\sim) = f_\pi; \tag{4}$$

$$(f \wedge g)_\pi = f_\pi \wedge g_\pi, (f \wedge g)^{\#} = f^{\#} \vee g^{\#} \left[\text{where } f^{\#} = (f_\pi \vee f_\pi^\sim)^\sim\right]; \tag{5}$$

$$f_\pi^\sim = (f_\pi \vee f^{\#})^\sim, f^{\sim\#} = f^{\#}; \tag{6}$$

$$(f * g)_\pi = f_\pi, (f * g)^{\#} = (f_\pi)^\sim \wedge (g_\pi^\sim)^\sim; \tag{7}$$

$$f_\pi * f^{\#} = f; \tag{8}$$

Therefore, A is an A^*-algebra. □

Theorem 2.5. Cayley theorem for A*-algebras (part II): *Every A^*-algebra is isomorphic to a Guard* - algebra on some set X.*

Proof. Let A be an A^*-algebra. $B = \{a_\pi / a \in A\} \Rightarrow B$ is a Boolean algebra. For every $a \in A$, define $f_a : B \times B \times B \to B$ by $f_a(x,y,z) = (a_\pi x) \vee (a_\pi^\sim y) \vee (a^{\#} z)$ for all $x, y, z \in B$ (where juxtaposition represents $\wedge$ i.e., $a_\pi x = a_\pi \wedge x$ etc.,) $\Rightarrow f_a \in Ter(B)$, for all $a \in A$.

Define, $f_a \wedge f_b = f_{a \wedge b}, (f_a)^\sim = f_{a^\sim}, f_a \vee f_b = f_{a \vee b}, f_a * f_b = f_{a*b}, \pi_1 = f_1, \pi_2 = f_0, \pi_3 = f_2$. It is easy to see that $\{f_a / a \in A\}$ satisfies 2.4 (i) to (iv).

$$(v) f_a(f_b(x,y,z), f_b(y,y,z), z) = f_a(b_\pi x \vee b_\pi^\sim y \vee b^\# z, b_\pi y \vee b_\pi^\sim y \vee b^\# z, z) =$$
$$= a_\pi(b_\pi x \vee b_\pi^\sim y \vee b^\# z) \vee (a_\pi^\sim(b_\pi y \vee b_\pi^\sim y \vee b^\# z) \vee a^\# z =$$
$$= a_\pi b_\pi x \vee a_\pi b_\pi^\sim y \vee a_\pi b^\# z \vee a_\pi^\sim b_\pi y \vee a_\pi^\sim b_\pi^\sim y \vee a_\pi^\sim b^\# z \vee a^\# z =$$
$$= a_\pi b_\pi x \vee a_\pi b_\pi^\sim y \vee a_\pi^\sim b_\pi y \vee a_\pi^\sim b_\pi^\sim y \vee (a_\pi \vee a_\pi^\sim) b^\# z \vee a^\# z =$$
$$= a_\pi b_\pi x \vee a_\pi b_\pi^\sim y \vee a_\pi^\sim b_\pi y \vee a_\pi^\sim b_\pi^\sim y \vee (a^{\#\sim} b^\# \vee a^\#) z =$$
$$= a_\pi b_\pi x \vee a_\pi b_\pi^\sim y \vee a_\pi^\sim b_\pi y \vee a_\pi^\sim b_\pi^\sim y \vee (a^\# \vee b^\#) z =$$
$$= a_\pi b_\pi x \vee (a_\pi b_\pi^\sim \vee a_\pi^\sim b_\pi \vee a_\pi^\sim b_\pi^\sim) y \vee (a^\# \vee b^\#) z =$$
$$= (a \wedge b)_\pi x \vee (a \wedge b)_\pi^\sim y \vee (a \wedge b)^\# z = f_{a \wedge b}(x,y,z) = (f_a \wedge f_b)(x,y,z).$$

$$(vi)(f_a * f_b) = f_{a*b}$$
$$f_{a*b}(x,y,z) = (a*b)_\pi x \vee (a*b)_\pi^\sim y \vee (a*b)^\# z =$$
$$= a_\pi x \vee (a_\pi)^\sim b_\pi^\sim y \vee (a_\pi)^\sim (b_\pi^\sim)^\sim z =$$
$$= a_\pi x \vee (a_\pi^\sim \vee a^\#) b_\pi^\sim y \vee (a_\pi^\sim \vee a^\pi)(b_\pi^\sim)^\sim z =$$
$$= a_\pi x \vee (a_\pi^\sim b_\pi^\sim y \vee a^\# b_\pi^\sim y \vee a_\pi^\sim b_{\pi^\sim}^\sim z \vee a^\# b_{\pi^\sim}^\sim z) =$$
$$= a_\pi x \vee a_\pi^\sim b_\pi^\sim y \vee a^\# b_\pi^\sim y \vee a_\pi^\sim (b_\pi \vee b^\#) z \vee a^\# (b_\pi \vee b^\#) z =$$
$$= a_\pi x \vee a_\pi^\sim b_\pi^\sim y \vee a^\# b_\pi^\sim y \vee a_\pi^\sim b_\pi z \vee a_\pi^\sim b^\# z \vee a^\# b_\pi z \vee a^\# b^\# z =$$
$$= a_\pi x \vee a_\pi^\sim (b_\pi z \vee b_\pi^\sim y \vee b^\# z) \vee a^\# (b_\pi z \vee b_\pi^\sim y \vee b^\# z) =$$
$$= f_a(x, f_b(z,y,z), f_b(z,y,z)).$$

Define $f : A \to Ter(B)$ by $f(a) = f_a$ for all $a \in A$. Clearly, f is a homomorphism. Suppose $a, b \in A$ and $f(a) = f(b)$ that is $f_a = f_b$. Then, $f_a(x,y,z) = f_b(x,y,z)$ for all $(x,y,z) \in B \times B \times B \Rightarrow a_\pi x \vee a_\pi^\sim y \vee a^\# z = b_\pi x \vee b_\pi^\sim y \vee b^\# z$, for all $(x,y,z) \in B \times B \times B$. If $x = 1, y = 0, z = 0$, then, $a_\pi = b_\pi$. If $x = 0, y = 0, z = 1$, then $a^\# = b^\#$. Therefore, $a = b$. Therefore, $f : A \to Ter\,(B)$ is an A^*-isomorphism. □

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